Feynman Rules and Renormalization of the Haag Series and Retarded Functions

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Graph rules are formulated, analogous to the Feynman rules, for the iterated solution of the Yang-Feldman equations ("Haag series") and for the retarded functions of the field operators. As a result of a renormalization procedure of a Bogoliubov-Parasiuk-Hepp type the retarded functions are correctly defined for any (finite) order of the perturbation expansion.

1. Introduction

The covariant graph technique and the renormalization procedure in a Bogoliubov-Parasiuk-Hepp fashion (BPH) [1, 2, 3], developed in this paper, originally arose to answer the needs of a new version of the quasipotential approach to the two-body problem, using the retarded functions as basic formalism.

The retarded (r-) functions are vacuum expectation values of the retarded (R-) products. The R-products are introduced in the theory by the following formal definition (in the simpliest case of a single, scalar, hermitian field $A(x)$, associated with spinless particles of mass m):

$$(1.1)\qquad R(A(x)A(x_1)\dots A(x_n))=R(x, x_1,\dots, x_n)$$

$$=\sum_p \theta(x^0-x^0_{i_1})\dots\theta(x^0_{i_{n-1}}-x^0_{i_n})[\dots[A(x), A(x_{i_1})],\dots, A(x_{i_n})],$$

$$R(\boldsymbol{A}(x))=A(x)$$

where $\sum_p$ is a summation over all permutations of $\{1,\dots, n\}$.

Within the framework of Wightman's field theory and Lehmann-Symanzik-Zimmermann's asymptotic condition (WLSZ field theory) the R-products are defined more precisely by a set of axioms [4, (i — vii)][1]. These axiomatic conditions do not define the R-products uniquely. It is easy to see that (1.1) is a possible realization of axioms [4, (i — vii)] in terms of tempered distributions over the subspace $S_0(\mathbf{R}^{4(n+1)})$ of Schwartz's test functions which "vanish strongy"[2] at points of at least one pair x_i, x_j, $i\neq j$ of equal arguments: $x_i=x_j$

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[1] The existence of the R-products obeying [4, (i — vii)] as a corollary from the WLSZ axioms has been proved so far in "formal power series sense", i. e. in each finite order of the perturbation expansion. The admission that WLSZ axioms as well as any other suggestion are true in "formal power series-sense" will be sufficient everywhere in our context.

[2] A function is said to "vanish strongly" at some point, if at this point it vanishes together with all its derivatives.

In case of a theory, involving at least one fermion field $\Psi(x)$ the definition of the R-products is modified [5].

The r-functions of the field operators are introduced with the relation:

$$r(x, x_1, \ldots, x_n) = i^n \langle 0 \mid R(x, x_1, \ldots, x_n) \mid 0 \rangle. \tag{1.2}$$

Our aim is to find a constructive realization of the R-products and r-functions over Schwartz's space S, suited for numerical perturbation calculations. We consider the case of Q. E. D. In section 2 we give a perturbative construction of the R-products and r-functions as tempered distributions in S'_0 in terms of R-diagrams, making a heuristic use of the iterated Yang-Feldman equations:

$$\begin{aligned} \Psi(x) &= \Psi^{\text{in}}(x) + e \int d^4y S_r(x-y)\gamma_\mu A^\mu(y)\Psi(y) \\ \overline{\Psi}(x) &= \overline{\Psi}^{\text{in}}(x) + e \int d^4y \overline{\Psi}(y) A^\mu(y)\gamma_\mu \overline{S}_r(x-y) \\ A_\mu(x) &= A^{\text{in}}_\mu(x) - e \int d^4y D^{\mu\nu}_r(x-y)\overline{\Psi}(y)\gamma_\nu\Psi(y), \end{aligned} \tag{1.3}$$

$$\begin{aligned} S_r(x) &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{(\hat{P}+m)}{m^2-p^2-ip^0 0}, \quad \overline{S}_r(x) = S_a(-x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{(\hat{P}+m)}{m^2-p^2+ip^0 0} \\ \hat{p} &= p^\mu\gamma_\mu, \quad D^{\mu\nu}_r(x) = -g^{\mu\nu} \int d^4p e^{-ipx}[(2\pi)^4 p^2 + ip^0 0]^{-1}. \end{aligned} \tag{1.4}$$

After the formal iteration of (1.3) we compare the result (in each perturbation order) with the corresponding Haag series[1]:

$$\begin{aligned} R(\ldots\Psi(x)\ldots\overline{\Psi}(y)\ldots A^\mu(z)\ldots) &= (-i)^s \sum_{f,\,b=0}^{\infty} \frac{1}{f!\,b!\,(f+f')!} \int \prod_1^{f} du_i \\ &\times \prod_1^{f+f'} dv_i \prod_1^{b} dw_i \, r^{\text{amp}}(\ldots\Psi(x)\ldots\overline{\Psi}(y)\ldots A^\mu(z)\ldots; \\ &\Psi(u_1)\ldots\Psi(u_f)\overline{\Psi}(v_1)\ldots\overline{\Psi}(v_{f+f'})A^{\mu_1}(w_1)\ldots A^{\mu_b}(w_b)) \\ &\times :\overline{\Psi}^{\text{in}}(u_1)\ldots\overline{\Psi}^{\text{in}}(u_f)\Psi^{\text{in}}(v_1)\ldots\Psi^{\text{in}}(v_{f+f'})A^{\mu_1}_{\text{in}}(w_1)\ldots A^{\mu_b}_{\text{in}}(w_b): \end{aligned} \tag{1.5}$$

where $s = f_0 + \overline{f}_0 + b_0$, $f' = |f_0 - f'_0|$; $f_0, \overline{f}_0, b_0$ is the number of fermion, antifermion ane photon fields respectively. We have used the following notation for the "amputated" r-functions [4]:

$$r^{\text{amp}}(x, x_1, \ldots, x_s; \; y_1, \ldots, y_n) = \left(\prod_{j=1}^{n} K_{y_j}\right) r(x, x_1, \ldots, x_s, y_1, \ldots, y_n), \tag{1.6}$$

$K_y = \Box_y$, $i\overrightarrow{\partial}_y - m$, $i\overleftarrow{\partial}_y + m$ for $A^\mu(y)$, $\Psi(y)$, $\overline{\Psi}(y)$ resp. We shall also need the "partially" (1.6′) and "totally" "amputated" r-functions [4]:

$$r(x, x_1, \ldots, \overbrace{x_j, \ldots, x_{j+k}}^{\text{amp}}, \ldots, x_n) = K_{x_j} \ldots K_{x_{j+k}} r(x, x_1, \ldots, x_j, \ldots, x_{j+k}, x_n), \tag{1.6′}$$

$$r^{\text{Amp}}(x, x_1, \ldots, x_n) = \left(\prod_{j=0}^{n} K_{x_j}\right) r(x, x_1, \ldots, x_n). \tag{1.6″}$$

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[1] According to Glaser-Lehmann-Zimmermann's reconstruction theorem [6] the Haag series (1.5) is strongly convergent if there exists a set of functions $r^{\text{amp}}(x; x_1, \ldots, x_n)$ satisfying the conditions that follow from (1.2), (1.6) and the axioms [4, (i — vii)].

After an appropriate reordering, the Haag series (1.5) acquires the following perturbation expansion form:

(1.7)
$$R(\varphi_1(x_1)\dots\varphi_s(x_s))$$
$$=\sum_{n=1}^{\infty}\frac{i^n}{n!}\int\prod_{j=1}^{n}d^4yR(\varphi_1^{\text{in}}(x_1)\dots\varphi_s^{\text{in}}(x_s)\mathcal{L}^{\text{in}}(y_1)\dots\mathcal{L}^{\text{in}}(y_n))$$

where $\varphi_i(x_i)$ denotes any of $A^\mu(x_i)$, $\Psi(x_i)$, $\overline{\Psi}(x_i)$ and

$$\mathcal{L}^{\text{in}}(y)=e:\overline{\Psi}_{\text{in}}(y)\gamma_\mu\Psi_{\text{in}}(y)A^\mu_{\text{in}}(y):$$

In section 3 we give an algorithm for extending the R- and r-distributions from S_0' to S'. The same problem was solved in Steinmann's analytic approach [4]. In it the r-functions are found to exist as recursive solution of Glaser-Lehmann-Zimmermann's equation [4, (2.36)] in S'. Despite of its logical simplicity this method is unsuited for numerical calculations. In this sense our work presenting a graph technique for the R-products (r-functions), analogous to the conventional Feynman rules, is a step forward.

Section 2. R-Diagrams

In this section we shall consider a covariant graph technique for the R-products and r-functions — R-diagrams (cf. [7]). It is prompted by the formal iteration of the Yang-Feldman equations (1.3). For $A^\mu(x)$, the following iterated expansion (up to the first order of the coupling constant) is obtained:

$$A^\mu(x)=A^\mu_{\text{in}}(x)-e\int d^4yD_r^{\mu\nu}(x-y):\overline{\Psi}^{\text{in}}(y)\gamma_\nu\Psi^{\text{in}}(y):$$

The comparison of this expansion with the corresponding Haag series (1.5), (1.7) provides the basis of introducing a technique similar to the Feynman rules. Table 1 represents the R-graph elements and their analytic expressions.

Everywhere in the context the Fourier transformation is applied in the form:

$$\tilde{f}(p_1,\dots,p_l)=\int\prod_{i=1}^{l}\left[du_i(2\pi)^{-\frac{1}{2}}\right]e^{i\Sigma p_ju_j}f(u_1,\dots,u_l)$$

which leads to an additional factor $(2\pi)^{-2n-C}$ in momentum space, n — order of the diagram, C — the number of closed loops.

Rules for construction of R-diagrams from the graph elements in Table 1:

(2.A) There are no disconnected diagrams;

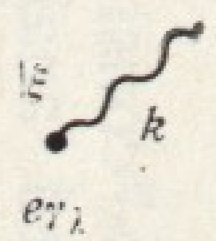
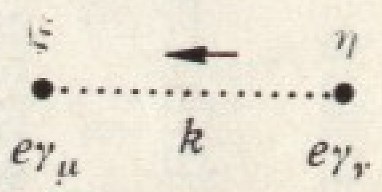
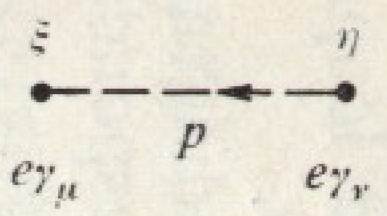
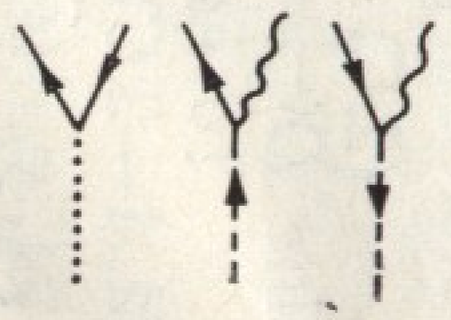
(2.B) Each fermion "path" (a polygon of fermion lines) is continuous and one-way directed. Two fermion paths can be connected either by means of a retarded propagator $D_r^{\mu\nu}$ or by one of the contraction functions $D^{(-)\mu\nu}$, $S^{(\pm)}$

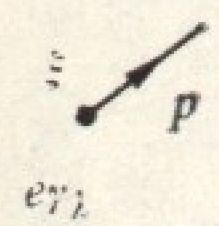
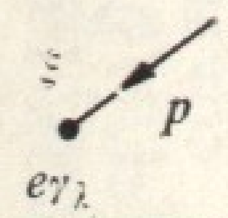
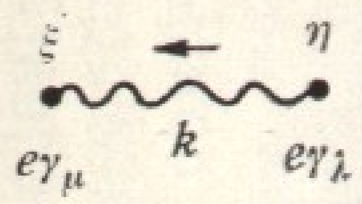
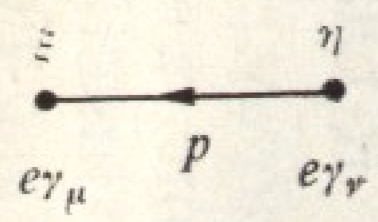
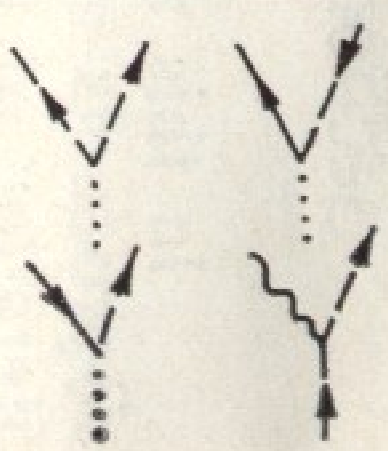
(2.C) Closed loops are formed only by means of at least one contraction line $D^{(-)\mu\nu}$, $S^{(\pm)}$.

To each R-diagram of n internal vertices constructed in agreement with (2.A—2.C) corresponds (Table 1) an n-ordered analytic expression $G_R(x;y_1,\dots,y_n)$ ($\tilde{G}_R(p;p_1,\dots,p_n)$), obeying the rules:

(2.D) All external lines correspond to normal product (Wick monomial) in G_R;

Tabel 1
Rules for the R-diagrams

Graph element	Analytic expression	
	configur. space	momentum space
	external photon line	
	$A^{\lambda}_{\text{in}}(\xi)$	$A^{\lambda}_{\text{in}}(k)$
	directed retardedphoton propagator	
	$-D^{\mu\nu}_{\text{r}}(\xi-\eta)$	$\dfrac{g^{\mu\nu}}{k^2+ik^00}$
	retarded (anti) fermion propagator	
	$S_{\text{r}}(\xi-\eta)$	$\dfrac{\hat{p}+m}{m^2-p^2-ip^00}$
	$\overline{S}_{\text{r}}(\xi-\eta)$	$\dfrac{\hat{p}+m}{m^2-p^2+ip^00}$
	external vertices	
	$e\gamma_{\lambda}$	$\delta(p_1+p_2+k)\, e\gamma_{\lambda}$

raph ment	Analytic expression	
	configur. space	momentum space
	external cutgoing fermion line	
	$\overline{\Psi}^{\text{in}}(\xi)$	$\overline{\Psi}^{\text{in}}(p)$
	external incoming fermion line	
	$\Psi^{\text{in}}(\xi)$	$\Psi^{\text{in}}(p)$
	directed contraction photon line	
	$iD^{(-)\mu\nu}(\xi-\eta)$	$-g^{\mu\nu}\theta(k^0)\delta(k^2)$
	dirccte anti)fermion cuntraction line	
	$\pm\frac{1}{i}\, S^{(\pm)}(\xi-\eta)$	$\mp\theta(\mp p^0)(\hat{p}+m \times\delta(p^2-m^2)$
	internal vertices	
	$e\gamma_{\lambda}$	$\delta(p_1+p_2+k)\, e\gamma_{\lambda}$

(2.E) The terms in G_R are ordered following a direction opposite to that of the fermion path (Fig. 1). On each vertex, together with the $e\gamma_\lambda$ factor, we also write down the contributions of the subdiagrams "hanged" at this vertex (blocks 1, 2 in Fig. 1) which may consist only of one or two external lines;

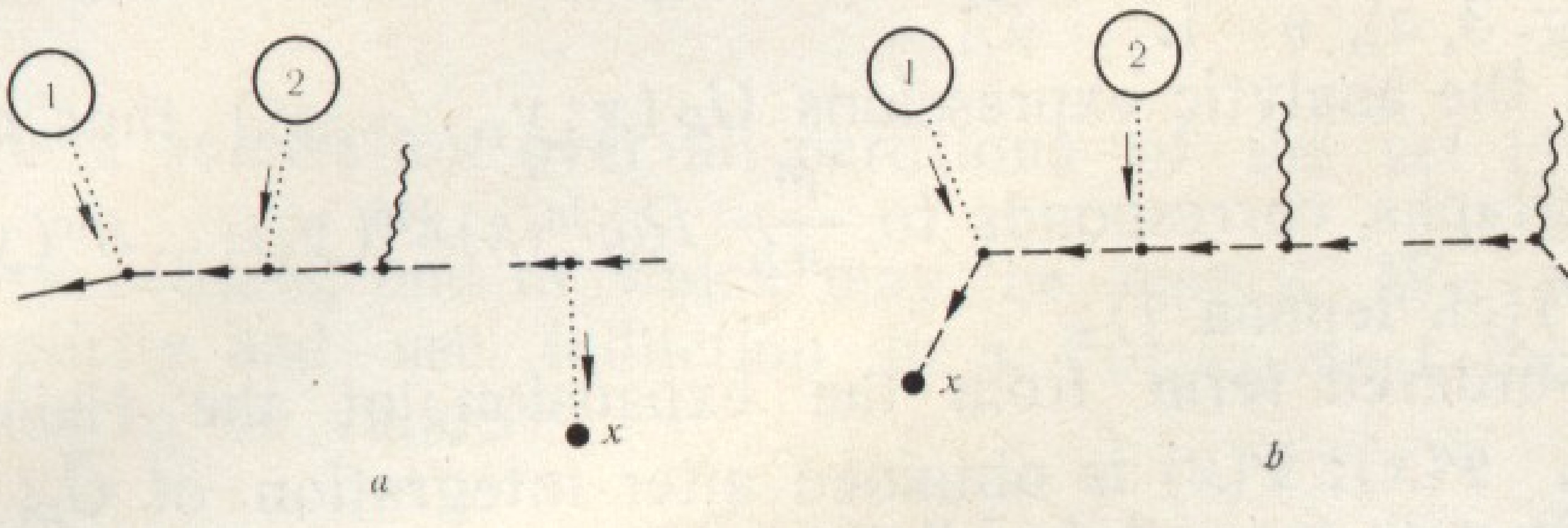


Fig. 1. Terms ordering in G_R

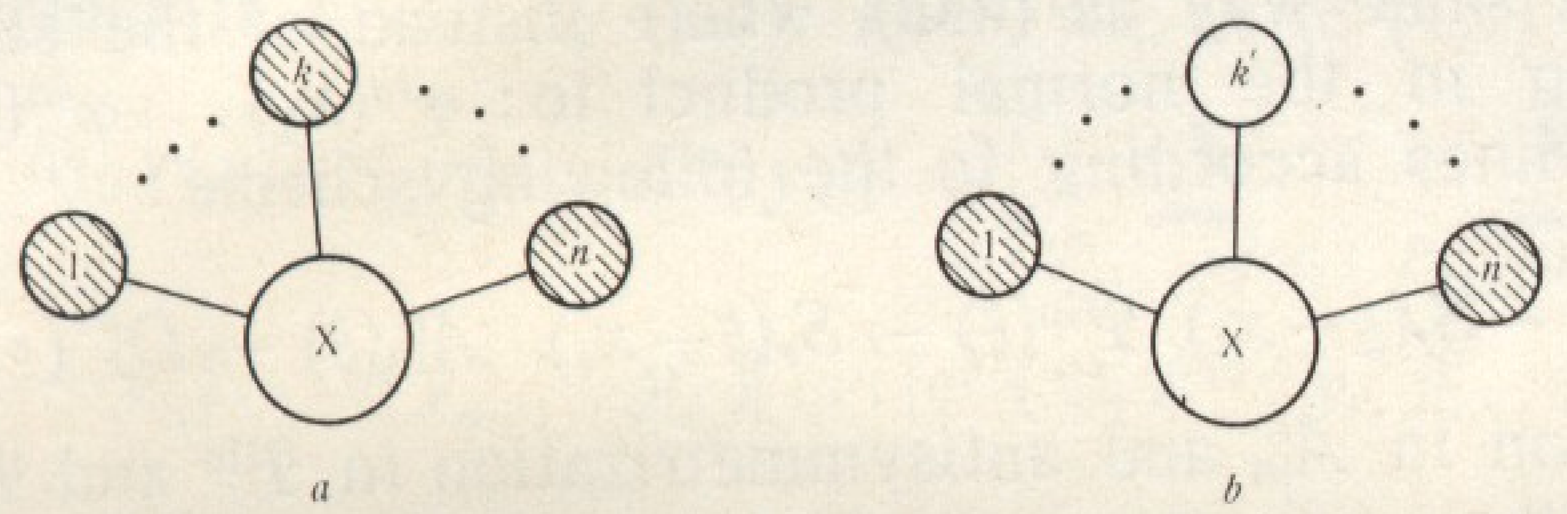


Fig. 2, *a*, *b*

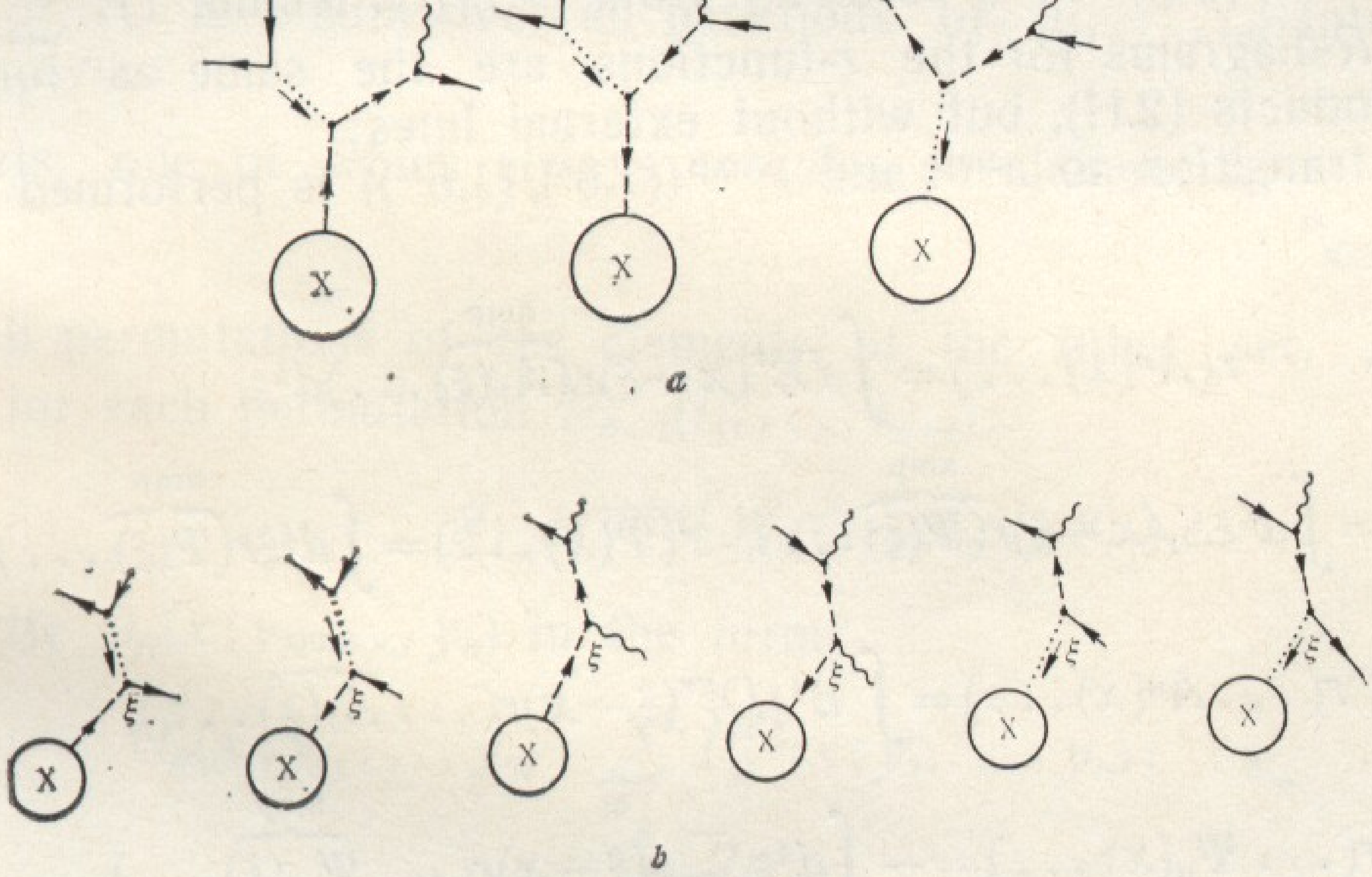


Fig. 3, *a*, *b*. Decomposition of a given R-graph into "current blocks" in order to find its combinatorial factor η. (The *X*-blocks contain the *x*-vertex)

(2.F) The combinatorial factor η of a given R-diagram, i. e., the numerical factor, with which it is present in the set of diagrams for the R-product, is defined recursively: $\eta=(n-1)!\sum_{k=1}^{n}\eta_k$ (Fig. 2, *a*), where η_k is the combinatorial factor of the diagram in (Fig. 2, *b*) in which the k'-block derives from the k-block in Fig. 2, *a*, with $\Psi^{\text{in}}(\xi)$, $\overline{\Psi^{\text{in}}}(\xi)$, $A^{\mu}_{in}(\xi)$ standing instead of the corresponding "current terms":

$$S_r(\xi-\eta):\hat{A}^{in}(\eta)\Psi^{in}(\eta):,\quad :\overline{\Psi}^{in}(\eta)\hat{A}^{in}(\eta):\overline{S}_r(\xi-\eta),\ D_r^{\mu\nu}(\xi-\eta):\overline{\Psi}^{in}(\eta)\gamma_\nu\Psi^{in}(\eta):$$

The blocks $1,\ldots,k,\ldots n$ in Fig. 2, *a* are of the type shown in Fig. 3, *a*, whereas the k'-block is of the type shown in Fig. 3, *b*. If the diagram cannot be decomposed in the form shown in Fig. 2, *a* into a X-block and "hanged" "current" blocks (Fig. 3, *a*), $\eta=1$.

The sum of the analytic expressions $G_{R_i}(x;y_1,\ldots,y_n)$ thus obtained from all n-ordered R-graphs, corresponds to $\frac{i^n}{n!}R(\varphi^{in}(x)\mathscr{L}^{in}(y_1)\ldots\mathscr{L}^{in}(y_n))$ as a corollary from (1.7) (cf. lemma 1);

(2.G) The n-ordered term from the expansion of the Heinsenberg field operators $A^\mu(x)$, $\overline{\Psi}(x)$; $\Psi(x)$ is obtained after integration of $\widetilde{G}_R(p;p_1,\ldots,p_n)$ $(G_R(x;y_1,\ldots,y_n))$ with respect to all internal momenta (all internal vertices);

(2.H) The n-ordered term from the R-product expansion (1.7) for $s>1$ is constructed in the same way as (2.G), where instead of the subset of external lines corresponding in the normal product to: $\varphi^{in}(x_1)\ldots\varphi^{in}(x_s)$: we place a subset of "blind" lines according to the following scheme:

$$\Psi^{in}(\xi)\to-\overline{S}_r(\xi-x_i)\ \overline{\Psi}^{in}(\xi)\to S_r(\xi-x_i),\ A^\mu_{in}(\xi)\to D_r^{\mu\nu}(\xi-x_i)$$

afrer symmetrization in A^μ_{in} and antisymmetrization in Ψ^{in} and $\overline{\Psi}^{in}$.

The sum of all n-ordered analytic expressions $G_{R_i}(x_1,\ldots,x_s;y_1,\ldots,y_n)$, $s>1$, with a fixed configuration of "blind" lines corresponds to $\frac{i^n}{n!}R(\varphi^{in}(x_1)\ldots\varphi^{in}(x_s)\mathscr{L}^{in}(y_1)\ldots\mathscr{L}^{in}(y_n))$ as a corollary from (1.7) (cf. lemma 1);

(2.I) The R-diagrams for the r-functions are the same as for the corresponding R-products (2.H), but without external lines;

(2.J) The transition to r^{amp} and r^{Amp} ((1.6′), (1.6″)) is performed using the formulas:

$$r(A^\mu(x)\ldots)=\int D_r^{\mu\nu}(x-\xi)r(\overbrace{A_\nu(\xi)}^{amp}\ldots)d^4\xi,$$

$$r(\Psi(x)\ldots)=\int d^4\xi S_r(x-\xi)r(\overbrace{\Psi(\xi)}^{amp}\ldots),\quad r(\overline{\Psi}(x)\ldots)=\int d^4\xi r(\overbrace{\overline{\Psi}(\xi)}^{amp}\ldots)\overline{S}_r(x-\xi),$$

$$r(\ldots A^\mu(x)\ldots)=\int d^4\xi D_r^{\mu\nu}(\xi-x)r(\ldots\overbrace{A_\nu(\xi)}^{amp}\ldots),$$

$$r(\ldots\Psi_\alpha(x)\ldots)=-\int d^4\xi\overline{S}_{r\alpha\alpha'}(\xi-x)r(\ldots\overbrace{\Psi_{\alpha'}(\xi)}^{amp}\ldots),$$

$$r(\ldots\overline{\Psi}_\alpha(x)\ldots)=\int d^4\xi r(\ldots\overbrace{\overline{\Psi}_{\alpha'}(\xi)}^{amp}\ldots)S_{r\alpha\alpha'}(\xi-x).$$

We shall prove the following lemma giving the correspondence between the analytic expressions of the R-graphs and the R-products:

L e m m a 1. Let $G_R(x_1,\ldots,x_s;y_1,\ldots,y_n)$ be the analytic expression for the set of all the n-ordered R-graphs with a fixed configuration of "blind" lines constructed accordfng to Table 1 and conditions (2.*A* — 2.*J*). Then $G_R(x_1,\ldots,x_s;y_1,\ldots,y_n)$ is of the form (1.1), for $\varphi^{in}(x_i)$, $i=1,\ldots,s$ and $\mathscr{L}^{in}(y_j)$, $j=1,\ldots,n$, and therefore satisfies the axioms [4, [i — vii)] for $R(\varphi^{in}(x_1)\ldots\varphi^{in}(x_s)\mathscr{L}^{in}(y_1)\ldots\mathscr{L}^{in}(y_n))$ over the subspace S_0 $(\mathbf{R}^{4(n+s)})$.

Proof: As a result of the construction described above, the axiomatic conditions [4, (i, iv, vi, vii)], except the R-unitarity (v), are satisfied over S_0. Therefore we have to check the validity of [4, (v)]:

$$R(x, y, X) - R(y, x, X) - \sum_{L,R} [R(x, X_L), R(y, X_R)]$$

where $\sum_{L,R}$ is a summation over all partitions of the set $\{1,\ldots,n\}$ into two complementary subsets, one of which may be empty. We consider the case $s=1$ for simplicity and use induction in n. For $\sigma \leq n-1$ the following identity is assumed to be valid:

$$G_R(x; y_1, \ldots, y_\sigma) = \frac{i^\sigma}{\sigma!} R(\varphi^{\text{in}}(x)\mathcal{L}^{\text{in}}(y_1)\ldots\mathcal{L}^{\text{in}}(y_\sigma))$$

where the R-product is given by (1.1). Then we write (1.1) in the form:

$$\frac{i^n}{n!} R(\varphi^{\text{in}}(x)\mathcal{L}^{\text{in}}(y_1)\ldots\mathcal{L}^{\text{in}}(y_n)) = \frac{i}{n} \sum_{P_{n-1}=fix,\,1} \sum_{P_{n-1}(i_1 \ldots i_{n-1})}$$

$$\times \theta(x^\circ - y^\circ_{i_1}) \ldots \theta(y^\circ_{n-1} - y^\circ_{i_n})[\ldots[\varphi^{\text{in}}(x), \mathcal{L}^{\text{in}}(y_{i_1})], \ldots, \mathcal{L}^{\text{in}}(y_{i_n})]$$

$$= \frac{i}{n} \sum'_{P_{n-1}=fix,\,1} [G_R(x; y_{i_1}, \ldots, y_{i_{n-1}}), \mathcal{L}^{\text{in}}(y_{in})](-1)^{P(i_1 \ldots i_{n-1})}$$

where $\sum_{P_{n-1}=fix,\,1}$ is a sum over all partitions of $\{i_1, \ldots, i_n\}$ into two complementary sets, one of which always contains an element i_n, $\sum_{P_{n-1}(i_1,\ldots i_{n-1})}$ is a sum over all permutations of the elements of the other set, „′" in $\sum'$ denotes that for each permutation $P_{n-1}(i_1, \ldots, i_{n-1})$:

(2.2) $$y^\circ_{i_n} = \min(y^\circ_{i_1}, \ldots, y^\circ_{in-1}, y^\circ_{i_n}).$$

We write $G_R(x; y_1, \ldots, y_n)$ in the form

(2.3) $$G_R(x; y_1, \ldots, y_n) = \sum_{W_n} \Gamma^{W_n}_R(x; y_1, \ldots, y_n) : \quad :_{W_n}$$

where the sum runs over all possible Wick monomials: $:_{W_n}$ of Ψ^{in}, $\overline{\Psi}^{\text{in}}$, A^{in}_μ in $G_R(x; y_1, \ldots, y_n)$, and $\Gamma^{W_n}_R(x; y_1, \ldots, y_n)$ are the coefficient functions. We insert (2.3) in (2.1) and afterwards expand the comutator according to Wick's theorem. Then it is easy to check that the only non-vanishing terms are those with one fermion, two and three arbitrary contractions and, due to (2.2), they are:

$$\sim S_r : \ : W_n,\ S_a : \ : W_n,\ S_r D^{\mu\nu}_r : \ :_{W_n},\ S_a D^{\mu\nu}_r : \ :_{W_n}.$$

Remembering that we have to (anti-)symmetrize in y_{i_n} we see that (2.1) yields exactly $G_R(x; y_1, \ldots, y_n)$. Therefore [4, (v)] is also valid.

Section 3. Renormalization Procedure

The result of lemma 1 indicates that the graph technique described in the previous section yields the R-product (r-functions) as tempered distributions over the subspace S_0. In this section we shall consider the problem of their renormalization in its two conventional aspects (a), b)):

a) Linear, continuous extension of R- and r-distributions from S_0' to S'. According to Hahn-Banch's theorem this procedure is not unique;

b) Removement of the arbitrariness by imposing additional requirements:

1. Requirements, resulting from the symmetries in the theory;
2. Requirement that the theory describes particles of physical mass;
3. Requirement for maximal scaling degree (minimal power increase in momentum space).

We shall show that one possible way to solve this problem for the R-products (r-functions) is a renormalization procedure in a BPH fashion [1—3]. From the viewpoint of the canonical Lagrangian approach this renormalization procedure is equivalent to the subtraction of appropriate quasilocal terms vanishing over S_0. This "subtraction" is abmissible due to the non-unique definition of the R-products (r-functions) in case of at least one pair of equial arguments.

Let Γ_R be a given R-graph. Its index of "superficial divergence" $\omega(\Gamma_R)$ is defined as usual [1]. It is easy to prove the identity: $\omega(\Gamma_R)=4-3F/2-B$, where F, B are the numbers of external fermion and photon lines respectively (F, m include the line ending in X).

We call $\gamma\subset\Gamma_R$ a "subgraph" of Γ_R if γ is a connected set of lines and vertices of Γ_R including at least one retarded propagator. If $\omega(\Gamma_R)\geqq 0$ for $\gamma\subset\Gamma_R$ the corresponding coefficient functions are not correctly defined in S' as a limit of their regularized expressions (e. g. by a Pauli-Villars regularization [1]). This situation is similar to the one with Feynman graphs [1, 2].

We call Γ_R "superficially divergent" if there exists at least one $\gamma\subseteqq\Gamma_R$, for which $\omega(\gamma)\geqq 0$. All sorts of "superficially divergent" R-graphs are shown

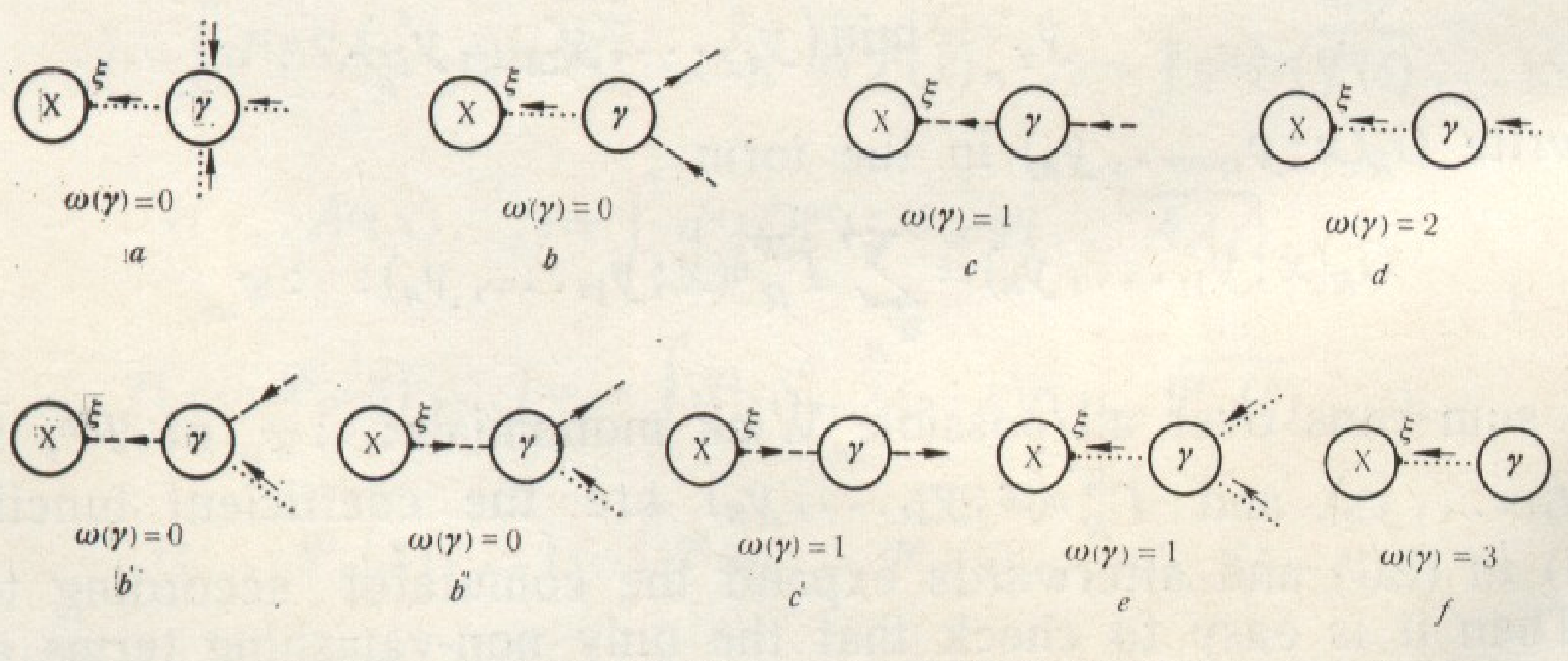


Fig. 4. All types of "superficially divergent" R-graphs

in Fig. 4. Their complete analogy to the types of "superficially divergent" Feynman graphs is obvious.

Due to the charge conjugation properties of $A^{\mu}(x)$ the r-functions of odd number photon fields vanish:

$$r(A^{\mu}(x)A^{\mu_1}(x_1)\dots A^{\mu_{n2}}(x_{2n}))\equiv 0.$$

Therefore the total contribution of all diagrams $4, e$, $4, f$ in Fig. 4 vanishes. This fact corresponds to Furry's theorem (e. g. in [1]). Lines going out of each generalized vertex γ (excluding those towards ξ) are of an arbitrary type.

One can easily be convinced, following the ideas of [1] that Ward-Takahashi type of identities are satisfied by the unrenormalized coefficient functions of the R-diagrams (i. e. over S_0).

In the same way as in [1, 2, 3] we introduce "$M\varepsilon$" regularization as a mapping:

$$\Gamma_R^{W_n}(x, x_1, \ldots, x_n) \longrightarrow \Gamma_R^{W_n, M_\varepsilon}(x, x_1, \ldots, x_n). \tag{3.1}$$

$\Gamma_R^{W_n, M_\varepsilon}$ is $\Gamma_R^{W_n}$ in which the retarded and contraction functions are replaced by their Pauli-Villars regularized expressions [1]. The mapping (3. 1) is equivalent to using in (1.1) $\mathcal{L}^{\text{in}}_{M_\varepsilon}(x)$ instead of $\mathcal{L}^{in}(x)$. $\mathcal{L}^{\text{in}}_{M_\varepsilon}(x)$ yields the "$M\varepsilon$" regularized Green's and contraction functions.

Obviously $R_{M_\varepsilon}(\varphi^{\text{in}}(x)\mathcal{L}^{\text{in}}(x_1) \ldots \mathcal{L}^{\text{in}}(x_n)) = R(\varphi^{\text{in}}(x)\, \mathcal{L}^{\text{in}}_{M_\varepsilon})(x_1) \ldots \mathcal{L}^{\text{in}}_{M_\varepsilon}(x_n))$ satisfies the axiomatic conditions [4, (i — vii)]. One can directly infer that the Ward-Takahashi identities are also valid for the regularized expressions $R_{M_\varepsilon}(\varphi^{\text{in}}(x)\mathcal{L}^{\text{in}}(x_1) \ldots \mathcal{L}^{\text{in}}(x_n))$ for each finite n.

As in [3] we introduce the following definition:

D e f i n i t i o n 1. Let $\mathcal{E}$ is a mapping:

$$\Gamma_R^{W_{n+s}}(x, x_1, \ldots, x_s, y_1, \ldots, y_n) \longrightarrow \mathcal{E}(\Gamma_R^{W_{n+s}}(x, x_1, \ldots, x_s, y_1, \ldots, y_n)).$$

$\mathcal{E}$ is a renormalization if $\sum\limits_{W_{n+s}} \mathcal{E}(\Gamma_R^{W_{n+s}}(x, x_1, \ldots, x_s; y_1, \ldots, y_n)\colon \ :_{W_{n+s}}$ satisfies all conditions (4, (i — vii)).

The most general solution for the set of operator products:

$T(x_1, \ldots, x_n)$, $\bar{T}(x_1, \ldots, x_n)$ (T-products), defined axiomatically, (cf. [3]), has the form [1]:

$$T'(x_1, \ldots, x_n) = T(\mathcal{L}^{\text{in}}(x_1) \ldots \mathcal{L}^{\text{in}})(x_n)) + \sum_{r=1}^{n-1} \sum_p T(L_{s_1}(x_1^p \ldots x_{s_1}^p \ldots L_{s_r}(\ldots x_n^p)) + L_n(x_1 \ldots x_n) \tag{3.2}$$

where $\sum\limits_p$ is a sum over all partitions of $\{1, \ldots, n\}$ into r complementary subsets each of which has $s_1, \ldots, s_r$ elements, $\sum\limits_i s_i = n$, $T(\mathcal{L}^{\text{in}}(x_1) \ldots \mathcal{L}^{\text{in}}(x_n))$ is a particular solution. The operators $L_s(x_1, \ldots, x_s)$ satisfy the conditions a)—f):

a) $L_s(x_1, \ldots, x_s) \in S'(\mathbf{R}^{4s})$;

b) $L_s(x_1, \ldots, x_s)$ are symmetric in all photon and antisymmetric in all fermion arguments;

c) $L_1(x) = \mathcal{L}^{\text{in}}(x)$;

d) $L_s(\Lambda(x_1 + a, \ldots, \Lambda x_s + a) = U(a, \Lambda) L_s(x_1, \ldots, x_s) U(a, \Lambda)$;

e) $\operatorname{supp} L_s(x_1, \ldots, x_s) = \{x_1, \ldots, x_s) \mid x_1 = x_2 = \cdots = x_s\}$;

f) $i^{s-1} L_s(x_1, \ldots, x_s)$ is hermitian.

There is one-to-one correspondence between the R-products and T-products [3, theorem 2.3]. Hence the most general solution $R'(x, x_1, \ldots, x_n)$ satisfying conditions [4, (i — vii)] acquires the form:

$$R'(x, x_1, \ldots, x_n) = R(\varphi^{\text{in}}(x)\mathcal{L}^{\text{in}}(x_1) \ldots \mathcal{L}^{\text{in}}(x_n)) + R(\varphi^{\text{in}}(x) L_n(x_1, \ldots, x_n)) \tag{3.3}$$

$$+\sum_{r=1}^{n-1} R(\varphi^{\mathrm{in}}(x) L_{s_1}(x_1^p \ldots x_{s_1}^p) \ldots L_{s_r}(\ldots x_n^p).$$

D e f i n i t i o n 2. (Subtraction procedure). We consider the following representation for the Pauli-Villars regularized expressions $\mathfrak{R}^{M\varepsilon}(\Gamma)$:

$$\mathfrak{R}^{M\varepsilon}(\Gamma)=\bar{\mathfrak{R}}^{M\varepsilon}(\Gamma)+\Lambda^{M\varepsilon}(\Gamma)=\delta(\Sigma p_i)\chi^{M\varepsilon}(\Gamma),\tag{3.4}$$

$$\bar{\mathfrak{R}}^{M\varepsilon}(\Gamma)=\sum_p\left\{\prod_{j=1}^{k(p)}\Lambda_{n_j}^{M_\varepsilon}(\gamma_j^p)\right\}\prod_{\mathrm{conn}}\Lambda_*^{M_\varepsilon}$$

where Γ denotes the coefficient function of the one-particle irreducible [1, 2] amputated R-graphs (i. e. graphs which are deprived of their initial $D_r^{\mu\nu}$, S_r, $\bar{S}_r$ lines — cf. (1.6')), $\sum_p$ runs over all partitions $\{\gamma_\gamma\}$, $j=1,\ldots, k(p)>1$ of γ into subgraphs, $\prod_{\mathrm{conn}}\Lambda_*^{M_\varepsilon}$ accounts for the contribution of all S_r, $\bar{S}_r$, $S^{(\pm)}$, D_r, $D^{(-}$ which connect different $\gamma_j^p \ldots \Lambda_{n_j}^{M_\varepsilon}(\gamma_j^p)$ are defined recursively:

$$\Lambda_n^{M\varepsilon}(\gamma)=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{if } \gamma \text{ is one-particle irreducible,}\\ -t^{d(\gamma)}\bar{\mathfrak{R}}^{M\varepsilon}(\gamma)+\bar{\Lambda}_n^{M\varepsilon}(\gamma) & \text{otherwise.}\end{cases}\tag{3.5}$$

$\bar{\Lambda}_n^{M\varepsilon}(\gamma)$ are generalized vertex parts [3]. They stay finite after the double limit $\lim_{\varepsilon\to 0}\lim_{M\to\infty}\bar{\Lambda}_n^{M\varepsilon}(\gamma)$. Their presence is a result of the "renormalization extension" of Γ. $t^{d(\gamma_j)}$ means taking the Taylor series of $\chi^{M\varepsilon}(p_1,\ldots,p_{n_j})$ around the origin up to the order $d(\gamma_j)$: $t^{d(\gamma_j)}\bar{\mathfrak{R}}^{M\varepsilon}(\gamma_j)=\delta(\Sigma p_j)t^{d(\gamma_j)}\chi(p_1,\ldots,p_j)$, $t^{d(\gamma)}\equiv 0$ for $\omega(\gamma)<0$.

The following two suggestions for the regularized coefficient functions of the R-graphs are valid:

L e m m a 2. Let the double limit $\lim_{\varepsilon\to 0}\lim_{M\to\infty}\mathcal{E}^{M\varepsilon}(\Gamma)=\mathcal{E}(\Gamma)$ exists for $\mathfrak{R}^{M\varepsilon}(\Gamma)$ constructed according to Definition 2. Then $\mathcal{E}(\Gamma)$ is a renormalization.

P r o o f: We consider

$$L_s^{M\varepsilon}(x_1,\ldots,x_s)=\sum_W\Lambda_{s,W}^{M_\varepsilon}(x_1,\ldots,x_s):\quad :_W\tag{3.6}$$

where $\Lambda_{s,W}^{M_\varepsilon}$ have the form (3.5) with $\bar{\Lambda}_{s,W}$ satisfying conditions $a)-f)$. $\bar{\Lambda}_{s,W}$ look like:

$$\bar{\Lambda}_{s,W}(x_1,\ldots,x_n)=\sum_{|\alpha|\le|D|}c_\alpha\left\{\prod_{i=1}^{n=1}\partial^{\alpha_i}/\partial x_i^{\alpha_i}\delta(x_i-x_{i+1})=\bar{\Lambda}_{s,W}(\gamma)\right.\tag{3.7}$$

in which the choice $|D|\le\omega(\gamma)$ is made (see below). Then taking into account (3.3) (3.4) and (3.6) we obtain:

$$R'_{W_\varepsilon}(\varphi^{\mathrm{in}}(x)\mathcal{L}^{\mathrm{in}}(x_1)\ldots\mathcal{L}^{\mathrm{in}}(x_n))=\sum_{W_n}\mathcal{E}^{M\varepsilon}(\Gamma_R^{W_n}(x,x_1,\ldots,x_n)):\quad :_{W_n}$$

But $\lim_{\varepsilon\to 0}\lim_{M\to\infty}\mathcal{L}^{\mu\varepsilon}(\Gamma)=\mathcal{E}(\Gamma)$ exists and R' is the most general solution of the axioms [4, i — vii)] (cf. (3.3)). This proves lemma 2.

The choice $|D|\leq\omega(\gamma)$ in (3.7), resp. $d(\gamma)=\omega(\gamma)$ in Definition 2 is *a new constraint* on the possible class of extensions. It is also present in the T-product renormalization procedure as well as in Steinmann's analytic approach [4] and is called "demand for maximal scaling degree" [4] or, equivalently, "demand for minimal power increase ω in momentum space" [9]. If $\omega(\Gamma_R)<0$ we have $\omega=\omega(\Gamma_R)$ as a corollary from Weinberg's power counting theorem [10] and $\mathcal{R}^{M_\varepsilon}(\Gamma^W_{R}{}_n)\in\mathcal{O}_M(\mathbf{R}^{4n})$. The same is true in the case $\omega(\Gamma_R)\geqq 0$ which can be proved by induction, following the ideas of [9].

The proof that the limit $\lim\limits_{M\to\infty}\mathcal{R}^{M_\varepsilon}=\mathcal{R}^\varepsilon$ exists is performed in complete analogy with Zimmermann's one for the T-products [8] applying Γ-forest technique.

Generally we have $\mathcal{R}^\varepsilon(\Gamma^W_{R}{}_n)\in\mathcal{E}\{S'_\omega(\mathbf{R}^{4(n+1)})$ where:

$$S_\omega=\{\Phi(X),\ \Phi(X)\in C^\omega(\mathbf{R}^{4n}),\ \sup[1+\|X\|)^A|D^B_x\Phi(X)|]<\infty,\ B\leqq\omega,$$

(3.8) $$A - \text{non-negative integers},\ X=(x_1,\ldots,x_n)\}$$

$$\mathcal{F}(S_\omega)=\{\tilde\Phi(p),\ \tilde\Phi(p)\in D^\infty(\mathbf{R}^{4n}),\ \sup[(1+\|P\|)^\alpha|D^\beta_p\tilde\Phi(p)|]<\infty,$$

$\alpha\leq\omega$, β — non-negative integers, $P=(p_1,\ldots,p_n)\}$, $\omega=\omega(\Gamma^W_{R}{}_n)$.

In a manner similar to (1) one can easily show by induction the gauge invariance of the subtraction procedure (definition 2). $\mathcal{L}^{M_\varepsilon}(x)$ acquires its well known form [1, (30.44—45)].

In accordance to Table 1 and (2.A) — (2.J) we get the following expression:

(3.9) $$R\varphi^{\text{in}}(x)\mathcal{L}^{\text{in}}(x_1)\ldots\mathcal{L}^{\text{in}}))=\sum_{j=1}^{n}\frac{1}{i}\Delta_R(x-x_j)\frac{\delta}{\delta\varphi^{\text{in}}(x_j)}$$
$$\times\{R(\mathcal{L}^{\text{in}}(x_j)\mathcal{L}^{\text{in}}(x_1)\ldots\hat{\mathcal{L}}^{\text{in}}(x_j)\ldots\mathcal{L}^{\text{in}}(x_n))\}$$

with $\Delta_R=-D^{\mu\nu}_r$, S_r, $\bar S_r$. (3.9) enables us to apply Epstein-Glaser renormalization technique [9] (in its modification for massless theories [11]). We shall essentially use the identity:

$$R(\mathcal{L}^{\text{in}}(x_1)\mathcal{L}^{\text{in}}(x_2)\ldots\mathcal{L}^{\text{in}}(x_s))-A(\mathcal{L}^{\text{in}}(x_1)\mathcal{L}^{\text{in}}(x_2)\ldots\mathcal{L}^{\text{in}}(x_n))$$
(3.10) $$=\sum_p(-1)^k[\bar T(\mathcal{L}^{\text{in}}(x_{i_1})\ldots\mathcal{L}^{\text{in}}(x_{i_k})),\ T(\mathcal{L}^{\text{in}}(x_1)\mathcal{L}^{\text{in}}(x_{i_{k+1}})\ldots\mathcal{L}^{\text{in}}(x_{i_{s-1}}))]$$

where A denotes the advanced product [3, (2.11)]. Repeating the ideas of section 2 we can give a graph construction and a renormalization scheme for the A-products up to the limit $\varepsilon\to 0$. There is one-to-one correspondence between A- and T-products [3].

Obviously the Pauli-Villars regularized expressions R^{M_ε}, A^{M_ε}, T^{M_ε}, $\bar T^{M_\varepsilon}$ and their limit when $M\to\infty$ satisfy (3.10).

The main result of the present section may be summarized in the following:

Theorem. $\lim\limits_{\varepsilon\to 0}\mathcal{R}_\varepsilon(\Gamma^W_{R}{}_n)=\mathcal{R}^{\text{ren}}(\Gamma^W_{R}{}_n)$ exists in $S'_\omega(\mathbf{R}^{4(n+1)})$ (3.8) for $\omega\doteq\omega(\Gamma^W_{R}{}_n)$ and is equal to the renormalization (definition 1) $\mathcal{R}^{EG}(\Gamma^W_{R}{}_n)$ constructed by Epstein-Glaser renormalization procedure.

Proof: It is accomplished by induction. The case $n=1$ is trivial. For $\sigma\leqq n-1$ we assume that the limit:

$\lim\limits_{\varepsilon\to 0}R_\varepsilon(\varphi^{\text{in}}(x)\mathcal{L}^{\text{in}}(x_1)\ldots\mathcal{L}^{\text{in}}(x_\sigma))=R^{\text{ren}}(\varphi^{\text{in}}(x)\mathcal{L}^{\text{in}}(x_1)\ldots\mathcal{L}^{\text{in}}(x_\sigma))$ exists.

Then taking into account [3, theorem (2.3)] and the results in [9, 11] we find that the same is true for $\overset{(-)}{T}_\varepsilon(x, x_1, \ldots, x_n)$. Thus the existence of the limit $\varepsilon \to 0$ in the right-hand side of (3.10) is ensured for R_ε, A_ε, T_ε, $\overline{T}_\varepsilon$. By means of the so called "cutting procedure" [9] we recover:

$$R^{\mathrm{ren}}(\mathscr{L}^{\mathrm{in}}(x_1)\ldots\mathscr{L}^{\mathrm{in}}(x_n)) = \lim_{\varepsilon\to 0} R_\varepsilon(\mathscr{L}^{\mathrm{in}}(x_1)\ldots\mathscr{L}^{\mathrm{in}}(x_n))$$

in $S'_\omega(\mathbf{R}^{4n})$ (3.8), $\omega = \omega(\Gamma'^{W}_{R}{}_n)$. $\Gamma'^{W}_{R}{}_n$ are the corresponding coefficient functions. $\mathscr{R}^{\mathrm{ren}}(\Gamma'^{W}_{R}{}_n) = \lim_{\varepsilon\to 0} \mathscr{R}_\varepsilon(\Gamma'^{W}_{R}{}_n)$ are defined up to arbitrary $\widetilde{\Lambda}(\Gamma'^{W}_{R}{}_n)$ as in (3.7) with $|D| \leq \omega(\Gamma'^{W}_{R}{}_n)$. $\widetilde{\Lambda}(\Gamma'^{W}_{R}{}_n)$ may be incorporated in $\overline{\Lambda}$ from (3.5). In a similar way we can prove in any case the existence of appropriate $\overline{\Lambda}_n$ (3.5), such that:

$$R^{\mathrm{ren}}(\mathscr{L}^{\mathrm{in}}(x_1)\ldots\mathscr{L}^{\mathrm{in}}(x_n)) = R^{EG}(\mathscr{L}^{\mathrm{in}}(x_1)\ldots\mathscr{L}^{\mathrm{in}}(x_n)).$$

The transition to $R(\varphi^{\mathrm{in}}(x_1)\mathscr{L}^{\mathrm{in}}(x_2)\ldots\mathscr{L}^{\mathrm{in}}(x_n))$ is performed using (3.9). With the help of the "adiabatic norms technique" [11] we verify the existence of the adiabatic limit:

$$\lim_{\varepsilon\to 0} \langle \mathscr{E}(\overbrace{\Gamma'^{W}_{R^n}(\xi_1,\ldots,\xi_s;\zeta_1,\ldots,\zeta_n)}^{\mathrm{amp}}) \,|\, \Phi\langle \xi_1,\ldots,\xi_n;\varepsilon\zeta_1,\ldots,\varepsilon\zeta_n\rangle\rangle$$

$$(3.12) \qquad = \langle r^{\mathrm{Amp}}(\xi_1,\ldots,\xi_s) \,|\, \varphi(\xi_1,\ldots,\xi_s)\rangle, \quad \mathscr{E}(\Gamma) = \mathscr{R}^{EG}(\Gamma).$$

(3.12) implies the correctly defined integrals is $S'_\omega(\mathbf{R}^{4n})$

$$R_n(\varphi(x)\varphi_1(x_1)\ldots\varphi_s(x_s)) = \frac{i^n}{n!}\int d^4y_1\ldots d^4y_n R^{\mathrm{ren}}(\varphi^{\mathrm{in}}(x)\varphi_1^{\mathrm{in}}(x_1)\ldots\varphi_s^{\mathrm{in}}(x_s)$$
$$\times \mathscr{L}^{\mathrm{in}}(y_1)\ldots\mathscr{L}^{\mathrm{in}}(y_n))$$

$$r_n(\varphi(x)\varphi_1(x_1)\ldots\varphi_s(x_s)) = \frac{i^{n+s}}{n!}\int d^4y_1\ldots d^4y_n \langle 0 \,|\, R^{\mathrm{ren}}\varphi^{\mathrm{in}}(x)\varphi_1^{\mathrm{in}}(x_1)\ldots\varphi^{\mathrm{in}}(x_s)$$
$$\times \mathscr{L}^{\mathrm{in}}(y_1)\ldots\mathscr{L}^{\mathrm{in}}(y_n)) \,|\, 0\rangle.$$

Following [11] we accomplish in the familiar way the "mass and charge" renormalization and the renormalization of the two-point photon wave function. In order to avoid the "infrared catastrophe" it is necessary to write the renormalization of the fermion wave function in the form:

$$\partial/\partial\hat{p}\{r^{\mathrm{Amp}}_{2n}(\Psi\overline{\Psi})(\hat{p})\}\,|_{\,p^2=\mu^2} = 0, \quad \mu^2 \neq m^2.$$

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Правила Фейнмана и ренормализационная процедура для ряда Хаага и запаздывающих функций

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(Резюме)

В работе формулируются правила, аналогичные Фейнмановским, для итерационного решения уравнений Янга-Фельдмана („ряд Хаага“) и для запаздывающих функций Гейзенберговских операторов поля в квантовой электродинамике. В результате ренормализационной процедуры типа Боголюбова-Парасюка-Хеппа зепаздывающие функции опрецеляются корректно для каждого (конечного) порядка теории возмущения.